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Exact Wave Functions of Time-Dependent Hamiltonian Systems Involving Quadratic, Inverse Quadratic, and $(1/\hat{x})\hat{p} + \hat{p}(1/\hat{x})$ Terms

J. R. Choi[1]

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We use the dynamical invariant method to derive quantum-mechanical solution of time-dependent Hamiltonian system consisting quadratic potential, inverse quadratic potential, and $(1/\hat{x})\hat{p} + \hat{p}(1/\hat{x})$. The term in Hamiltonian containing $(1/\hat{x})\hat{p} + \hat{p}(1/\hat{x})$ gives the expression such as $(1/\hat{x})(\partial/\partial\hat{x})$ in coordinate space, which we can often meet in radial equation of quantum many body problem. The wave functions differed only a time-dependent phase factor from the eigenstates of the invariant operator $\hat{I}$ and expressed in terms of an associated Laguerre function.

KEY WORDS: invariant operator; wave function; $(1/\hat{x})\hat{p} + \hat{p}(1/\hat{x})$ term.

1. INTRODUCTION

The analysis and investigation of quantum harmonic oscillator contributed to the development of modern physics (Moshinsky and Smirnov, 1996). The quantum state of time-independent N-body problem with harmonical and inverse quadratic potential has been obtained by Calogero (1969a). He derived complete energy spectrum and all the corresponding eigenfunctions of the system. In this paper, we extend this method to the somewhat more complicated case where the coefficients of the Hamiltonian depend explicitly on time. The quantum solutions for time-dependent Hamiltonian systems have been studied in the literature for several decades (Choi, 2002; Lewis and Riesenfeld, 1969; Pedrosa *et al.*, 1997; Trifonov, 1999; Um *et al.*, 1996, 1998, 2001a,b, 2002a,b; Yeon *et al.*, 1993, 1994; Zhang *et al.*, 2002).

However, in most of the cases, the naive derivation of the quantum solutions for the time-dependent Hamiltonian system by separation of variables $\hat{x}$ and t in Schrödinger equation is not easy due to the complexity of the equation. One method to remedy this defect is the introduction of dynamical invariants. The dynamical

[1] Department of Electronic Physics, College of Natural Science, Hankuk University of Foreign Studies, Yongin 449–791, Korea; e-mail: choiardor@hanmail.net.

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invariants, firstly observed by Lewis in 1967 (Lewis, 1967), can be used powerfully to investigate the quantum state of the time-dependent Hamiltonian system. Lewis and Riesenfeld used the dynamical invariant method to derive the quantum-mechanical solution of a charged particle in a time-dependent electromagnetic field (Lewis and Riesenfeld, 1969). Afterwards, this method was developed further by several authors (Korsch, 1979; Ray and Reid, 1979).

The translation invariant model of N-body problem is observed by Calogero (1996). The quantum states of the time-dependent harmonic oscillator has been studied in the literatures (Yeon *et al.*, 1993, 1994). The susceptibility for identical atoms subjected to an external force with tail is analyzed quantum-mechanically (Um *et al.*, 1996).

In this paper, we will derive the exact quantum state of the time-dependent Hamiltonian systems involving quadratic, inverse quadratic, and $(1/\hat{x})\hat{p} + \hat{p}(1/\hat{x})$ terms using dynamical invariant method. Our model can be applied to solve the radial equation of time-dependent quantum many body system (Calogero, 1969a,b, 1971; Sutherland, 1971), the problem of noninteracting electrons that the effective mass varies as time goes by under the vector potential chosen by $\mathbf{A} = (By/2, Bx/2, 0)$ (J. R. Choi, unpublished; Dittrich *et al.*, 1998) and the radial equation for ordinary isotropic oscillator (Choi and Kweon, 2002; Shankar, 1979).

2. THE EIGENVALUE AND EIGENSTATE OF INVARIANT OPERATOR

Let us consider the time-dependent Hamiltonian

$$\hat{H}(\hat{x}, \hat{p}, t) = A(t)\hat{p}^2 + B(t)(\hat{x}\hat{p} + \hat{p}\hat{x}) + C(t)\left(\frac{1}{\hat{x}}\hat{p} + \hat{p}\frac{1}{\hat{x}}\right) + D(t)\hat{x}^2 + E(t)\frac{1}{\hat{x}^2}, \tag{1}$$

where $A(t), \ldots, E(t)$ are time-dependent coefficients that differentiable with respect to time, and $A(t) \neq 0$. Note that the term containing $(1/\hat{x})\hat{p} + \hat{p}(1/\hat{x})$ gives the expression containing $(1/\hat{x})(\partial/\partial\hat{x})$ in coordinate space that appears in radial equation of quantum many body problems (Calogero, 1969a,b, 1971; Sutherland, 1971). Here we would like to remark that as far as we know the time-dependent Hamiltonian containing $(1/\hat{x})\hat{p} + \hat{p}(1/\hat{x})$ such as Eq. (1) have not yet been treated in the literature.

The corresponding classical equation of motion can be derived using Hamilton's equation as

$$\ddot{\hat{x}} - \frac{\dot{A}}{A}\dot{\hat{x}} + 2\left(2AD + \frac{\dot{A}B}{A} - 2B^2 - \dot{B}\right)\hat{x} + 2\left(\frac{\dot{A}}{A}C - \dot{C}\right)\frac{1}{\hat{x}} + 4(C^2 - AE)\frac{1}{\hat{x}^3} = 0. \tag{2}$$

To facilitate the derivation of quantum state, let us introduce the invariant operator of the system. We can assume that the trial invariant operator has the form.

$$\hat{I}(\hat{x}, \hat{p}, \hat{t}) = \alpha(t)\hat{x}^2 + \beta(t)(\hat{x}\hat{p} + \hat{p}\hat{x}) + \gamma(t)\hat{p}^2 + \delta(t)\left(\frac{1}{\hat{x}}\hat{p} + p\frac{1}{\hat{x}}\right) + \eta(t)\frac{1}{\hat{x}^2} + \xi(t), \tag{3}$$

where $\alpha(t) - \zeta(t)$ are time-dependent coefficients which should be determined afterwards. Owing to its definition, the invariant operator Eq. (3) must satisfy the following relation.

$$\frac{d\hat{I}}{dt} = \frac{\partial \hat{I}}{\partial t} + \frac{1}{i\hbar}[\hat{I}, \hat{H}] = 0. \tag{4}$$

By inserting Eqs. (1) and (3) into Eq. (4), we can derive the following relations between coefficients

$$\dot{\alpha} = 4(\beta D - \alpha B), \tag{5}$$

$$\dot{\beta} = 2(\gamma D - \alpha A), \tag{6}$$

$$\dot{\gamma} = 4(\gamma B - \beta A), \tag{7}$$

$$\dot{\delta} = 4(\delta B - \beta C), \tag{8}$$

$$\dot{\eta} = 4(\eta B - \beta E), \tag{9}$$

$$\dot{\xi} = 4(\delta D - \alpha C), \tag{10}$$

$$\eta A = \gamma E, \tag{11}$$

$$\eta C = \delta E, \tag{12}$$

$$\delta A = \gamma C. \tag{13}$$

To simplify the problem, let us impose the following auxiliary condition (Trifonov, 1999)

$$\frac{E(t)}{A(t)} = \text{Constant}. \tag{14}$$

Then we can solve Eqs. (5)–(13) to give the explicit value of coefficients in invariant operator as

$$\alpha(t) = \frac{1}{4A^2}(2Bs(t) - \dot{s}(t))^2 + \frac{E}{A}\frac{1}{s^2(t)}, \tag{15}$$

$$\beta(t) = \frac{1}{2A}(2Bs^2(t) - s(t)\dot{s}(t)), \tag{16}$$

$$\gamma(t) = s^2(t), \tag{17}$$

$$\delta(t) = \frac{C}{A} s^2(t), \tag{18}$$

$$\eta(t) = \frac{E}{A} s^2(t), \tag{19}$$

$$\xi(t) = 4 \int_0^t \left[\frac{CD}{A} s^2(t) - \frac{C}{4A^2} (2Bs(t) - \dot{s}(t))^2 - \frac{CE}{A} \frac{1}{s^2(t)} \right] dt, \tag{20}$$

where $s(t)$ is some time-dependent classical solution of the following differential equation

$$\ddot{s}(t) - \frac{\dot{A}}{A} \dot{s}(t) + 2 \left(2AD + \frac{\dot{A}B}{A} - 2B^2 - \dot{B} \right) s(t) - 4AE \frac{1}{s^3(t)} = 0. \tag{21}$$

By inserting Eqs. (15)–(20) into Eq. (3), we can obtain the full expression of invariant operator as

$$\begin{aligned} \hat{I}(\hat{x}, \hat{p}, t) = {} & \left[\frac{1}{4A^2} (2Bs(t) - \dot{s}(t))^2 + \frac{E}{A} \frac{1}{s^2(t)} \right] \hat{x}^2 + \frac{1}{2A} (2Bs^2(t) \\ & - s(t)\dot{s}(t))(\hat{x}\hat{p} + \hat{p}\hat{x}) + s^2(t)\hat{p}^2 + \frac{C}{A} s^2(t) \left(\frac{1}{\hat{x}} \hat{p} + \hat{p} \frac{1}{\hat{x}} \right) \\ & + \frac{E}{A} s^2(t) \frac{1}{\hat{x}^2} + \xi(t). \end{aligned} \tag{22}$$

Let us denote the eigenvalue and eigenstate of the invariant operator $\hat{I}(\hat{x}, \hat{p}, t)$ as λ and $\phi(\hat{x}, t)$, respectively:

$$\hat{I}(\hat{x}, \hat{p}, t)\phi(\hat{x}, t) = \lambda \phi(\hat{x}, t). \tag{23}$$

Substitution of Eq. (22) into Eq. (23) and after some arrange, we obtain that

$$\left[\frac{\partial^2}{\partial \hat{x}^2} + \left(a \frac{1}{\hat{x}} - b\hat{x} \right) \frac{\partial}{\partial \hat{x}} - c\hat{x}^2 - d \frac{1}{\hat{x}^2} + \Lambda \right] \phi(\hat{x}, t) = 0, \tag{24}$$

where

$$a = \frac{2iC}{\hbar A}, \tag{25}$$

$$b = \frac{i}{\hbar A} \left(\frac{\dot{s}}{s} - 2B \right), \tag{26}$$

$$c = \frac{1}{\hbar^2 s^2} \left[\frac{1}{4A^2} (2Bs(t) - \dot{s}(t))^2 + \frac{E}{A} \frac{1}{s^2(t)} \right], \tag{27}$$

$$d = \frac{1}{\hbar A} \left(iC + \frac{E}{\hbar} \right), \tag{28}$$

$$\Lambda = \frac{i}{2\hbar A}\left(2B - \frac{\dot{s}}{s}\right) + \frac{1}{\hbar^2 s^2}(\lambda - \xi). \tag{29}$$

To solve Eq. (24), we introduce the variable $\hat{r}$ as

$$\hat{r} = \hat{x}^2, \tag{30}$$

and express ϕ as

$$\phi(\hat{r}, t) = \hat{r}^k e^{q\hat{r}} y(\hat{r}, t), \tag{31}$$

where k and q are given by

$$k = \frac{1}{4}\left(1 - a + \sqrt{(1-a)^2 + 4d}\right), \tag{32}$$

$$q = \frac{1}{4}\left(b - \sqrt{b^2 + 4c}\right). \tag{33}$$

By inserting Eq. (31) into Eq. (24) and after some rearrangement, we can obtain the differential equation

$$\hat{z}y''(\hat{z}, t) + \left(2k + \frac{1}{2}(1+a) - \hat{z}\right) y'(\hat{z}, t) + \frac{1}{2}\left[\frac{\lambda - \xi}{2\hbar}\sqrt{\frac{A}{E}} - m - l - 1\right] y(\hat{z}, t) = 0, \tag{34}$$

where

$$\hat{z} = \frac{1}{2}\sqrt{b^2 + 4c}\,\hat{r}, \tag{35}$$

$$m = \frac{1}{2}\sqrt{1 + \frac{4E}{\hbar^2 A}}, \tag{36}$$

$$l = \frac{Cs(\dot{s} - 2Bs)}{2\hbar A\sqrt{AE}}. \tag{37}$$

Because Eq. (34) satisfies associated Laguerre polynomial defined in Erdély (1953), we can express y as

$$y = L_n^m(\hat{z}), \tag{38}$$

where

$$n = \frac{1}{2}\left[\frac{\lambda - \xi}{2\hbar}\sqrt{\frac{A}{E}} - m - l - 1\right]. \tag{39}$$

From Eq. (39), we can see that the eigenvalue of the invariant operator can be represented as

$$\lambda = 2\hbar\sqrt{\frac{E}{A}}(2n + m + l + 1) + \xi. \tag{40}$$

Inserting Eqs. (32), (33), and (38) into Eq. (31), we obtain the nth eigenstate of the invariant operator as

$$\begin{aligned}\phi_n(\hat{x}, t) = &\left\{ \frac{2}{{}_{n+m}P_m} \left(\frac{1}{\hbar s^2}\sqrt{\frac{E}{A}} \right)^{(m+1)} \right\}^{1/2} \hat{x}^{m+[1-2iC/(\hbar A)]/2} \\ &\times \exp\left\{ -\frac{1}{4}\left[\frac{i}{\hbar A}\left(2B - \frac{\dot{s}}{s} \right) + \frac{2}{\hbar s^2}\sqrt{\frac{E}{A}} \right]\hat{x}^2 \right\} \\ &\times L_n^m\left(\frac{1}{\hbar s^2}\sqrt{\frac{E}{A}}\hat{x}^2 \right),\end{aligned} \tag{41}$$

where ${}_{n+m}P_m$ is permutation that defined as $(n+m)!/n!$.

3. WAVE FUNCTION

The wave function of the system is different from the eigenstate of invariant operator by only some time-dependent phase factor $\exp[i\theta_n(t)]$ (Lewis and Riesenfeld, 1969). Thus, we can represent the wave function as

$$\psi_n(\hat{x}, t) = \phi_n(\hat{x}, t)\exp[i\theta_n(t)]. \tag{42}$$

Equation (42) must satisfy the following Schrödinger equation

$$i\hbar\frac{\partial\psi_n(\hat{x}, t)}{\partial t} = \hat{H}(\hat{x}, \hat{p}, t)\psi_n(\hat{x}, t). \tag{43}$$

By inserting Eqs. (1) and (42) into Eq. (43), we can obtain that

$$\hbar\dot{\theta}_n(t) = \left\langle \phi_n \right| \left(i\hbar\frac{\partial}{\partial t} - \hat{H} \right) \left| \phi_n \right\rangle. \tag{44}$$

Phase $\theta_n(t)$ can be calculated by inserting Eq. (41) into Eq. (44) as

$$\theta_n(t) = -(2n + m + l + 1)\int_0^t \frac{\sqrt{A(t')E(t')}}{s^2(t')}\,dt' - \frac{1}{\hbar}\int_0^t \xi(t')\,dt'. \tag{45}$$

By substitution of Eqs. (41) and (45) into Eq. (42), we can obtain the full wave function as

$$\psi_n(\hat{x}, t) = \left\{ \frac{2}{{}_{n+m}P_m} \left(\frac{1}{\hbar s^2} \sqrt{\frac{E}{A}} \right)^{(m+1)} \right\}^{1/2} \hat{x}^{m+[1-2iC/(\hbar A)]/2}$$
$$\times \exp\left\{ -\frac{1}{4} \left[\frac{i}{\hbar A} \left(2B - \frac{\dot{s}}{s} \right) + \frac{2}{\hbar s^2} \sqrt{\frac{E}{A}} \right] \hat{x}^2 \right\} L_n^m \left(\frac{1}{\hbar s^2} \sqrt{\frac{E}{A}} \hat{x}^2 \right)$$
$$\times \exp\left[-i(2n + m + l + 1) \int_0^t \frac{\sqrt{A(t')E(t')}}{s^2(t')} \, dt' - \frac{i}{\hbar} \int_0^t \xi(t') \, dt' \right]. \tag{46}$$

Thus, we can confirm that the wave function that can be used to calculate the various quantummechanical expectation values is expressed in terms of associated Laguerre function. For $C(t) = 0$, Eq. (46) exactly reduces to that of Choi and Kweon (2002).

4. SUMMARY

We used dynamical invariant method to investigate the quantum solution of time-dependent Hamiltonian system involving quadratic, inverse quadratic, and $(1/\hat{x})\hat{p} + \hat{p}(1/\hat{x})$ terms. Note that the term in Hamiltonian containing $(1/\hat{x})\hat{p} + \hat{p}(1/\hat{x})$ gives the corresponding expression such as $(1/\hat{x})(\partial/\partial\hat{x})$ in coordinate space, which we can often meet in radial equation of quantum many body problem (Calogero, 1969a,b, 1971; Sutherland, 1971). Although, the connections between quantum solutions and classical solutions of the time-dependent harmonic oscillator with and without an inverse quadratic potential has been studied by various authors (Pedrosa *et al.*, 1997; Um *et al.*, 1996, 1998; Yeon *et al.*, 1993, 1994), the quantum solutions of time-dependent Hamiltonian containing $(1/\hat{x})\hat{p} + \hat{p}(1/\hat{x})$ such as Eq. (1) have not yet been solved in the literature as far as we know. To facilitate the derivation of quantum state, we introduced dynamical invariant operator, $\hat{I}$. We obtained the eigenvalue and eigenstate of invariant operator. We derived the exact wave function of the system using the fact that it is different from the eigenstate of invariant operator by only some time-dependent phase factor $\exp[i\theta_n(t)]$. The wave function of the system expressed in terms of associated Laguerre function and can be used to calculate the various quantum-mechanical expectation values. For $C(t) = 0$, we confirmed that our results agrees with that of previous papers (Choi and Kweon, 2002; Pedrosa *et al.*, 1997).

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